Classification of n-tone rows with Generalized Chord Diagrams

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Generalizations

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- We associate notes belonging to cosets for groups different from $\mathbb{Z}/2\mathbb{Z}$.

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- Classification obtained with generalized chord diagrams of a single type.
- Classification obtained with generalized chord diagrams of all types.

Example of generalized chord diagram

We consider the case $H = \mathbb{Z}/4\mathbb{Z} \cong 3\mathbb{Z}/12\mathbb{Z} < \mathbb{Z}/12\mathbb{Z}$
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Induced partition is

$$P_r^H = (\{0, 1, 3, 4\}, \{2, 9, 10, 11\}, \{5, 6, 7, 8\})$$

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- The row r induces an equivalence relation $\sim$ on $\mathbb{Z}/n\mathbb{Z}$, where $i \sim j \Leftrightarrow a_i - a_j \in H$.
- The relation $\sim$ divides $\mathbb{Z}/n\mathbb{Z}$ into d subsets of order m, which we define to be the *partition associated to the row r* and we write P_r^H.

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- For a given n-tone row r, the chord diagram associated to r with H as characteristic subgroup is the equivalence class of P_r^H in $\mathcal{R}^H/D_n$ and is written $\mathcal{C}_r^H$.

Group action on the set of n-tone rows

We are looking, given a subgroup H of $\mathbb{Z}/n\mathbb{Z}$, for a faithful action of a group Γ_H on the set S_n of n-tone rows with the following property: $\forall s, t \in S_n$

$$\exists \gamma \in \Gamma_H : \gamma(s) = t \Leftrightarrow \mathcal{C}_s^H = \mathcal{C}_t^H$$

Decomposition of the group Γ_H (1)

On the set S_n we have the following group actions:

- The action of the dihedral group D_n defined by the following:

$\forall r = (a_0, a_1, a_2, \ldots, a_{n-1})$, $\forall d \in D_n$

$$d \cdot r = (a_{d(0)}, a_{d(1)}, \ldots, a_{d(n-1)})$$

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Example: for $d = T_1$, $r = (0, 3, 4, 9, 6, 11, 2, 8, 5, 10, 7, 1)$

$$d \cdot r = (3, 4, 9, 6, 11, 2, 8, 5, 10, 7, 1, 0)$$

It is not the obvious action of the dihedral group.

Decomposition of the group Γ_H (2)

Having fixed a subgroup H, we have the action of the group

- $E_H = \{\sigma \in S_n |\ \sigma(x + H) = x + H\ \forall x \in \mathbb{Z}/n\mathbb{Z}\}$

defined by: $\forall r = (a_0, a_1, a_2, \ldots, a_{n-1})$, $\forall \sigma \in E_H$

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- $E_H \cong (S_m)^d$

Decomposition of the group Γ_H (3)

- We fix a set of representatives $\mathcal{J}$ for $\frac{\mathbb{Z}/n\mathbb{Z}}{H}$.
- We consider the transformations $\tau_{ij} \in S_n$, for $i \neq j \in \mathcal{J}$ defined by the composition of transpositions $(i+h, j+h)$ $\forall h \in H$.
- We call $F_H^{\mathcal{J}}$ the group generated by these transformations.
- $\tau_{ij} \cdot (a_0, a_1, \ldots, a_{(n-1)}) = (\tau_{ij}(a_0), \tau_{ij}(a_1), \ldots, \tau_{ij}(a_{(n-1)}))$

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- $F_H^{\mathcal{J}}$ depends on the choice of $\mathcal{J}$.
- $F_H^{\mathcal{J}} \cong S_d$

The group G_H

We define G_H to be the group generated by E_H and $F_H^{\mathcal{J}}$ (G_H does not depend on the choice of $\mathcal{J}$).

- $G_H \cong E_H \rtimes F_H^{\mathcal{J}}$
- $\forall r, s \in S_n \; P_r^H = P_s^H \Leftrightarrow \exists g \in G_H : g \cdot r = s$

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$$\Gamma_H =< D_n, G_H >= D_n \times G_H = D_n \times (E_H \rtimes F_H)$$

Number of series associated to a chord diagram

$$\forall r \in S_n \; |\Gamma_H| = |Orb(r)| \cdot |Stab(r)|$$

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$$|\{s \in S_n | \mathcal{C}_s^H = \mathcal{C}_r^H\}| = \frac{|\Gamma_H|}{|Symm(r)|}$$

$$|\Gamma_H| = |D_n| \cdot |E_H| \cdot |F_H| = 2n \cdot m!^d \cdot d!$$

Objective

We want a group Λ which acts on the set S_n with the property that, for all r and s in S_n, r and s belong to the same orbit if and only if they have the same generalized chord diagram for every subgroup of $\mathbb{Z}/n\mathbb{Z}$.

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- Λ is the intersection of the groups Γ_H, with H varying on the set of subgroups of $\mathbb{Z}/n\mathbb{Z}$.

Trivial subgroups of Λ

- The dihedral group D_n.
- The group of affine transformations $Aff(\mathbb{Z}/n\mathbb{Z})$, which is contained in G_H for any subgroup H.

We will see that the inclusion of the group $D_n \times Aff(\mathbb{Z}/n\mathbb{Z})$ in Λ is, in general, proper. However, in the case $n = 12$, the two groups coincide.

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It is sufficient su describe the group $\Omega = \bigcap G_H$.

Properties of the groups F_H

- For $H < I$ it is possible to choose representatives $\mathcal{J}$ for $\mathbb{Z}_n/H$ and $\mathcal{J}'$ for $\mathbb{Z}_n/I$ in such a way that $F_I^{\mathcal{J}'} < F_H^{\mathcal{J}}$.

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- To do so, having chosen the representatives $\mathcal{J}'$, we need to choose the representatives $\mathcal{J}$ in such a way that that the generators of $F_I^{\mathcal{J}'}$ act as a permutation on them ($F_I^{\mathcal{J}'}(\mathcal{J}) = \mathcal{J}$).

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- This can be done from a set $\mathcal{J}'$ of representatives of $\mathbb{Z}_n/I$ and defining the representatives $\mathcal{J}$ to be the sum of elements of $\mathcal{J}'$ with representatives of I/H.

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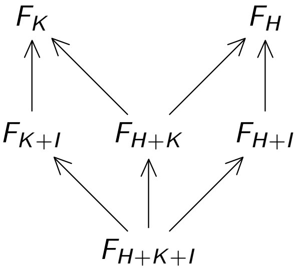
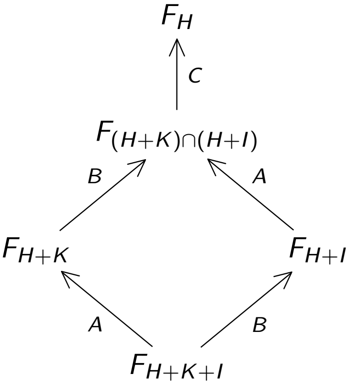
Example: $n = 12$, $H = \mathbb{Z}/2\mathbb{Z} < I = \mathbb{Z}/4\mathbb{Z}$, $\mathcal{J}' = \{0, 1, 2\}$ we choose $\{0, 9\}$ as representatives of I/H and we obtain $\mathcal{J} = \{0, 1, 2, 9, 10, 11\}$ by sum.
$\tau_{02}(\{0, 1, 2, 9, 10, 11\}) = \{9, 1, 11, 0, 10, 2\} = \mathcal{J}$

Choice of representatives (1)



We want representatives for all groups of the type $\mathbb{Z}_n / \sum H_i$ which give all inclusions between the groups $F_{\sum H_i}$.

Choice of representatives (2)



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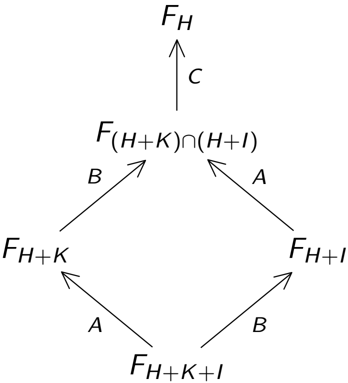

For $\mathbb{Z}_n/(H+K+I)$ we choose any set $\mathcal{J}$ of representatives.

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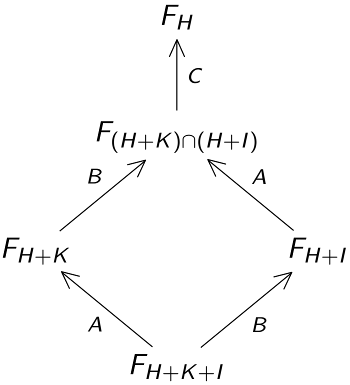
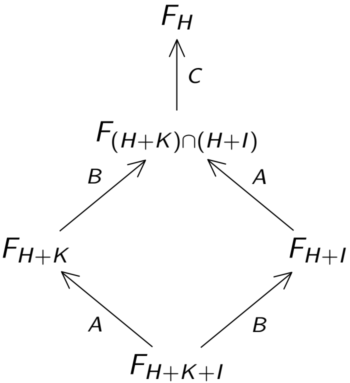


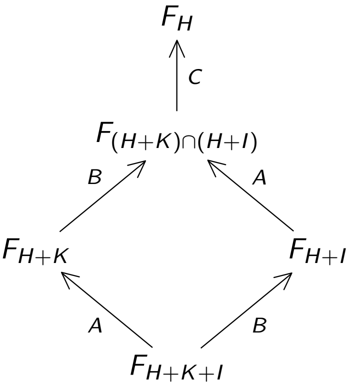
For $(H+K+I)/(H+K)$ we choose a set A of representatives for $I/I \cap (H+K)$.

Choice of representatives (2)



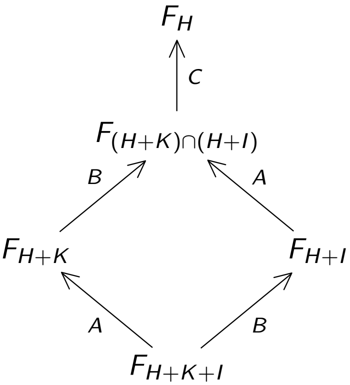
For $(H+K+I)/(H+I)$ we choose a set B of representatives for $K/K\cap(H+I)$.

Choice of representatives (2)



For $(H+I)/(H+K)\cap(H+I)$ we can choose A as set of representatives.

Choice of representatives (2)



For $(H+K)/(H+K)\cap(H+I)$ we can choose B as set of representatives.

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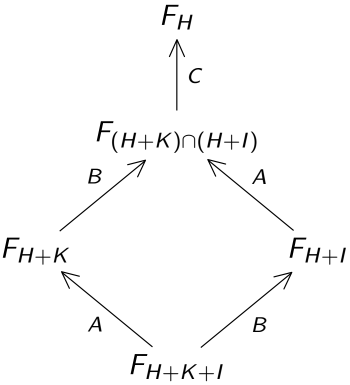

For $(H+K)\cap(H+I)/H$ we can choose a set C of representatives of $(K\cap I)/(K\cap I\cap H)$.

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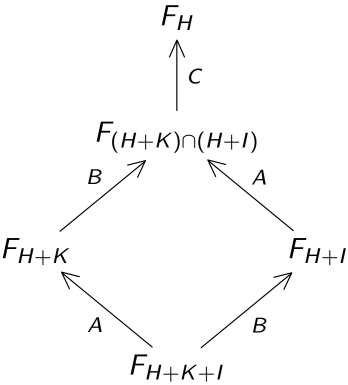

Well defined choice of representatives for $\mathbb{Z}_n/H$, and all inclusions are verified.

Decomposition of the group Ω (1)

- There exists a good choice of representatives which verifies all inclusions.
- With this choice, one can show that $F_{H+K} = F_H \cap F_K$
- $E_{H\cap K} = E_H \cap E_K$

We expect $\Omega = \bigcap G_H$ (in the case of four non trivial subgroups I, J, K, H) to be generated by subgroups of the form

- $E_{I\cap J\cap H\cap K}$
- $F_I \cap E_{J\cap H\cap K}$
- $F_{I+H} \cap E_{J\cap K}$
- $F_{I+H+J} \cap E_K$
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2. Compose it with elements from $F_{H+K} \cap E_I$, $F_{H+I} \cap E_K$, $F_{K+I} \cap E_I$ to obtain an element in $E_{H+K} \cap E_{H+I} \cap E_{K+I}$

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3. Compose it with elements from $F_H \cap E_{K\cap I}$, $F_K \cap E_{H\cap I}$, $F_I \cap E_{H\cap K}$ to obtain an element of $E_H \cap E_K \cap E_I = E_{H\cap K\cap I}$

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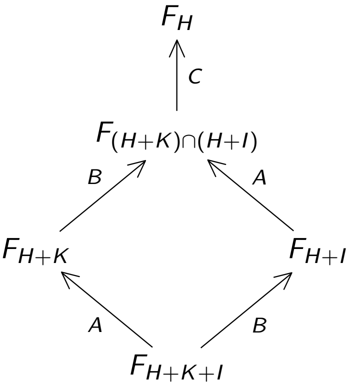
4. Compose it with an element of $E_{H \cap K \cap I}$ to obtain the identity.

Decomposition of the group Λ (3)

In the case of three non trivial subgroups H, I, K
$\Omega \cong E_{H\cap I\cap K} \rtimes F_I \cap E_{H\cap K} \rtimes F_H \cap E_{I\cap K} \rtimes F_K \cap E_{I\cap H} \rtimes$
$\rtimes F_{H+I} \cap E_K \rtimes F_{I+K} \cap E_H \rtimes F_{H+K} \cap E_I \rtimes F_{I+H+K}$

$$\Lambda = D_n \times \Omega$$

Descriptions of the subgroups



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- If $i = j_1 + a_1 + b_1 + c_1$, $l = j_2 + a_2 + b_2 + c_2$, $j_i \in \mathcal{J}$, $a_i \in A$, $b_i \in B$, $c_i \in C$ we require $j_1 = j_2$, $a_1 = a_2$, $b_1 = b_2$. We therefore have as many choices as the index d of $(H + K) \cap (H + I)$ in $\mathbb{Z}_n$.

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- For c_1 and c_2, we have as many choices as the order m of $(K \cap I)/(K \cap I \cap H)$.
- $F_H \cap E_{K \cap I} \cong (S_m)^d$, and has order $m!^d$

The case $n = 15$

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$Aff(\mathbb{Z}/15\mathbb{Z}) < \Omega$ is a proper inclusion.

The case $n = 12$

$I = \mathbb{Z}/2\mathbb{Z}$, $J = \mathbb{Z}/3\mathbb{Z}$, $H = \mathbb{Z}/4\mathbb{Z}$, $K = \mathbb{Z}/6\mathbb{Z}$
Of all the subgroups, the only non trivial are:

- $F_{I+J+K} \cap E_H = F_K \cap E_H$ with $2! = 2$ elements.
- $F_{I+H} \cap E_{J \cap K} = F_H \cap E_J$ with $3! = 6$ elements.
- $F_J \cap E_{I \cap H \cap K} = F_J \cap E_I$ with $2!^2 = 4$ elements.

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Ω has therefore $2 \cdot 6 \cdot 4 = 48$ elements, which is exactly the order of $Aff(\mathbb{Z}/12\mathbb{Z})$. Therefore the two groups coincide and we obtain

$$\Lambda \cong D_{12} \times Aff(\mathbb{Z}/12\mathbb{Z})$$