The sieve of Eratosthene and Johnson's problem on rhythmic canons

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- A music-theoretical problem of tiling the line
- Its reformulation in terms of polynomials and Diophantine equations (supposedly 325–409)
- Its numerical solution by an algorithm which resembles the sieve of Eratosthene (284–192 BC) for finding prime numbers.
- Computational aspects
- Practical application: *Eine kleine Mathmusik*
- Discussion and farther problems

1 Introduction: Rhythmic canons

Johnson (2001) intended to build a rhythmic canon using three rhythmic patterns of crotchets and quarter rests coded respectively by 1s and 0s:

Pattern number	Musical meaning	Progression of tones and rests
1	Theme	1 1 0 0 1
2	Theme in augmentation	1 0 1 0 0 0 0 0 1 0
3	Theme in double augmentation	1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0

Assumption 1 (No gap) *No rest occurs simultaneously in all voices.*

Assumption 2 (No double beat) *No tone occurs simultaneously in any of two voices.*

Voice	Pattern	Beat number														
number	number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	0	0	1										
2	1			1	1	0	0	1								
3	2						1	0	1	0	0	0	0	0	1	0
4	1									1	1	0	0	1		
5	1											1	1	0	0	1
Simultaneous onsets		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

2 Polynomial representation of rhythmic canons

Represent rhythmic patterns by polynomials with coefficients 0 or 1:

$$P = 1\,1\,0\,0\,1 \quad \longleftrightarrow \quad p(x) = 1 + 1x + 0x^2 + 0x^3 + 1x^4 \ .$$

If the pattern delays by 2 beats, multiply $p(x)$ by x^2:

$$P_2 = 0\,0\,1\,1\,0\,0\,1 \quad \longleftrightarrow \quad 0+0x+1x^2+1x^3+0x^4+0x^5+1x^6 = p(x)x^2 \ .$$

No shift corresponds to the multiplication of $p(x)$ by the unit 1 .
A superposition of rhythmic patterns corresponds to the sum of associated polynomials:

$$P + P_2 = 1\,1\,1\,1\,1\,0\,1 \quad \longleftrightarrow \quad p(x) + p(x)x^2 = p(x)(1 + x^2) \ .$$

A double beat results in a coefficient 2 instead of 1 for a single beat.
Multiple superpositions of $P \leftrightarrow p(x)$ with delays correspond to polynomial products $p(x)q(x)$, where $q(x)$ is associated with (multiple) time delays:

$$p(x)q(x), \quad \text{where} \quad q(x) = 1 + x^2 + x^8 + x^{10} \ .$$

Consider a rhythmic canon generated by a single rhythmic pattern $P \leftrightarrow p(x)$ with no augmentations. Associate voice delays with a polynomial $q(x)$. Assumptions 1–2 mean that

$$p(x)q(x) = I_n(x) = \sum_{i=0}^{n} x^n \ ,$$

where n is the sum of degrees of $p(x)$ and $q(x)$. In this case, the length of the canon is $n+1$ beats.

Proposition 1 (Existence and uniqueness of a simple rhythmic cano
A rhythmic canon generated by a single pattern $P \leftrightarrow p(x)$ can be $n+1$ beats long if and only if $I_n(x)$ is divisible by $p(x)$ and the coefficients $q(x)$ are either 0s, or 1s. If such a canon exists, it is unique to within permutation and union of voices.

"Unique to within permutation and union of voices" means that no new canon emerges if we (a) renumber the voices, or (b) reduce the number of voices by putting disjoint rhythmic patterns into the same voice.

3 Rhythmic canons with augmentation

Consider a rhythmic pattern

$$P_0 \quad \longleftrightarrow \quad p_0(x) = \sum_{i=1}^{k} a_i x^i \ .$$

Its jth augmentation P_j corresponds to the polynomial

$$P_j \quad \longleftrightarrow \quad p_j(x) = \sum_{i=0}^{k} a_i x^{2^j k} \ .$$

A rhythmic canon built from the 'theme' $P_0 \leftrightarrow p_0(x)$ and its two successive augmentations P_1 and P_2 must satisfy the polynomial equation

$$p_0(x)q_0(x) + p_1(x)q_1(x) + p_2(x)q_2(x) = I_n(x) \ , \tag{1}$$

where polynomial $q_j(x)$ is associated with entry delays of the jth augmentation. For our example: (1) for the following polynomials:

$$\begin{aligned} p_0(x) &= 1 + x + x^4 \ , & q_0(x) &= 1 + x^2 + x^8 + x^{10} \ , \\ p_1(x) &= 1 + x^2 + x^8 \ , & q_1(x) &= x^5 \ , \\ p_2(x) &= 1 + x^4 + x^{16} \ , & q_2(x) &= 0 \ , \\ I_n(x) &= 1 + x + \cdots + x^{14} \ . \end{aligned}$$

4 No analytical solution to Johnson's problem

Note that polynomials are 'generalized numbers':

- They include numbers as polynomials of degree 0.
- Addition, subtraction, multiplication, and division are defined for polynomials similarly to that for numbers.
- The division properties of polynomials are similar due to the unique factorization into irreducible polynomials, which are polynomial analogue of prime numbers.
- The polynomial classes inherit some properties of numbers which are used for their coefficients: one can consider integer coefficients, or rational coefficients, or real coefficients, etc.

From this standpoint, the equation (1) is a polynomial version of Diophantine equation

$$p_0q_0 + p_1q_1 + p_2q_2 = I$$

with positive integer coefficients p_0, p_1, p_2, I and to be solved in positive integers q_0, q_1, q_2. For instance, the Diophantine equation

$$5q_0 + 7q_1 = 100 \tag{2}$$

has two solutions, $(6, 10)$ and $(13, 5)$.

The existence of a general analytical solution (with a formula) to (1) would mean the existence of an analytical solution to the much more simple Diophantine equation (as for polynomials of degree 0). Since no solution to Diophantine equations is known, there is little chance to solve more general 'Diophantine equations' for polynomials.

By the way recall that Fermat (1601–1665) has formulated his Great Theorem as a margin note in Diophante's *Arithmetic* as a step towards the unsolvable general case.

5 The sieve of Eratosthene for Diophantine equations

The analogy of polynomials with numbers enables us to find solutions for practice-relevant short canons. While selecting rhythmic canons satisfying Assumptions 1 and 2, we use the idea of the sieve of Eratosthene for finding prime numbers.

The sieve of Eratosthene operates in several runs. Consider the first N positive integers.

1. Select 2 as prime. Sort out all numbers divisible by 2 by striking out every second number.
2. Select the first remaining number greater than 1 as prime, in this case 3. Sort out all numbers divisible by 3 by striking out every third number.
3. Select the first remaining number greater than 1 as prime, in this case 5. Sort out all numbers divisible by 5 by striking out every fifth number.

... Continue unless all N integers are stroke out.

A similar approach is applicable to Diophantine equations to sort out inappropriate candidates for solution. The main problem here is that each run turns out to be a branching process. For example, solutions to (2) can be selected by fixing $q_0 = 1, 2, \ldots$ and then attempting to reach 100 from $5q_0$ by 7-long steps.

Some additional restriction, significantly reducing the number of branches, are most helpful in operationalizing the process. In our example, $100 - 7q_1$ must be divisible by 5, reducing q_1 to 5 and 10, which implies the solutions required.

6 Coding convention

A good coding convention is often the ladder to success in combinatorics. An algorithm for enumerating canons must use as few parameters as possible.

Proposition 2 (Coding convention) *Under Assumptions 1 and 2, a rhythmic canon coded by a succession of entering rhythmic patterns is unique to within permutation and union of voices.*

7 The sieve of Erathosphen for rhythmic canons

Under our coding convention, a canon C is determined by a ternary number

$$C = \{\pi_1\pi_2\ldots\pi_k\}, \qquad \text{where} \quad \pi_k = 1, 2, 3 \ ,$$

which, being represented by rhythmic patterns or polynomials, satisfies Assumptions 1 and 2 (= equation (1)).

For sorting out inappropriate ternary numbers we use a kind of sieve of Eratosthene. The analogy is two-fold:

- If we consider an element (canon) then we delete the branch with its successors, which stems from this element.
- We always start with the first remaining element.

The Candidates for canon are to be collected in list C of candidates. The kth candidate is a ternary number $C[k]$, e.g. $C[k] = \{1121\}$.

The Selected canons, satisfying Assumptions 1– 2, is another list of ternary numbers S. For instance, the first selected canon is $S[1] = \{11211\}$.

Creating a new element of list C is appending either 1, 2, or 3 to the currently considered ternary number $C[k]$. The new element can be either rejected, or selected into list S, or farther retained in C as a candidate. In the latter case the new ternary number is appended to the end of list C. Since the ternary number currently processed is no longer needed, it is deleted. Therefore, the element currently processed is always the first in list C. Thus C is destroyed from the beginning (as 3000 elements are accumulated), appended to the end (at each iteration), and some elements of C are moved to S.

The list of selected canons has no repeats in the sense that no smaller canon is a part of a larger canon. Indeed, if a canon is accomplished then it is moved from the list of candidates to the selected list, leaving no descendants in list C. In other words, each selected canon is continuous, with the end of a rhythmic pattern in one voice occurring in the middle of a rhythmic pattern of some other voice.

The algorithm cannot miss any of canons, because it is based on generating ternary numbers with all branches.

8 Computing

The program has been written in the MATLAB (= MATrix LABoratory) C++-based computer programming environment for matrix and vector operations.

The program output is a LaTeX text file with the code of LaTeX-tables.

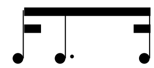
One can switch on and off the deleting the first elements of list C.

Computing at a PC with a Pentium 300MHz-processor.

Totally tested combinations	1260234
Maximum number of candidates for canon in memory	72996
Average number of candidates for canon in memory	42653
Found canons of length 15	1
Found canons of length 30	6
Found canons of length 45	20
Found canons of length 60	93
Found canons of length 75	348
Found canons of length 90	1460
Found canons of length 105	5759
Found canons of length 120	23502
Totally selected canons	700
Computation time, in seconds	7069

9 Example of application: *Eine kleine Mathmusik*

PREREQUISITES

Basic rhythmic pattern 1 1 0 0 1 = 
Also in augmentation and double augmentation.

Time : 5/16.

Basic melodic pattern: an ascending major third and ascending second, e.g. G,B,C.

Physical voice: unites several disjoint canon voices (a) to reduce the number of musicians and (b) to construct motives

Canon No. 1 of length 15 with 5 canon voices and 3 physical voices

V-ce	Patt.	Score		
1	1	1 1 0 0 1	. . . . .	. . . . .
2	1	. . 1 1 0	0 1 . . .	. . . . .
3	2	. . . . .	1 0 1 0 0	0 0 0 1 .
4	1	. . . . .	. . . 1 1	0 0 1 . .
5	1	. . . . .	. . . . .	1 1 0 0 1

Piece: a sequence of canons arranged for a woodwind sextet (the least number of physical voices possible).

Form structure: $1 + 1 + 2 + \cdots$

Canon ends as cadances: canons are separated by rests 1/16 – 3/16 as rhythmic stops (cadances).

Style: neobaroque (major-minor harmony, rules of polyphony, and usual tonal development within a piece).

Musical form: Sonata form (with two themes).

Variation of canons: new ends or extensions, e.g.

$$1121\boxed{1} \quad \longrightarrow \quad 1121\boxed{3\ 3\ 1\ 1\ 2\ 1}$$

MUSICAL FORM

- **Exposition**, measures 1–54

  1st theme in G, measures 1–6: the unique shortest canon (twice)

$$\underbrace{11211}_{15\text{ beats} = 3\text{ bars},\ G\to D} \quad + \quad \underbrace{11211}_{15\text{ beats} = 3\text{ bars},\ C\to G} \quad .$$

  1st transition, measures 7–18: two closest variations of the theme

$$\underbrace{1121\boxed{3\ 3\ 1\ 1\ 2\ 1}}_{30\text{ beats} = 6\text{ bars},\ C\to C_7} + \quad \underbrace{1121\boxed{332222}}_{30\text{ beats} = 6\text{ bars},\ F\to F_{6/9}}$$

  with polyphonic indices 4 and 6, respectively.

  2nd transition, measures 19–30: the next variation of the theme (twice)

$$\underbrace{11\boxed{3\ 1\ 2\ 1\ 1\ 2\ 1\ 1}}_{30\text{ beats} = 6\text{ bars},\ d_m\to A_7} + \underbrace{11\boxed{3\ 1\ 2\ 1\ 1\ 2\ 1\ 1}}_{30\text{ beats} = 6\text{ bars},\ d_m\to F_6} \quad .$$

  2nd theme in D, measures 31–42:

$$\underbrace{112\boxed{22223321113121 1211}}_{60\text{ beats} = 12\text{ bars},\ D\to F\sharp_7}$$

  Extension of the 2nd theme, measures 43–54:

$$\underbrace{11\boxed{3121131211213311 21}}_{60\text{ beats} = 12\text{ bars},\ E\to A_+}$$

- **Development**, measures 55–120

  1st theme in D, measures 55–60.

  Variation of the 1st transition, measures 61–84: periodic canon with 3 full periods of length 30 beats (= 6 bars),

$$\underbrace{1}_{G}\ \underbrace{1222233211}_{D,B_7,a_{m6},E_7}\ \underbrace{1222233211}_{G,E_7,d_{m6},A_7}\ \underbrace{1222233211}_{C,A_7,g_{m6},D_7}\ \underbrace{121332222}_{F\to G_7}$$

  Variation of the 2nd transition, measures 85–96

$$\underbrace{1131211\boxed{31211312 11211}}_{60\text{ beats} = 12\text{ bars},\ c_m\to A\flat}$$

2nd theme in C, measures 97–108, $C \rightarrow E_7$

Extension of the 2nd theme in D, measures 109–120, $D \rightarrow G_+$.

- **Recapitulation,** measures 121–150

  1st theme in C, measures 121–126

  Variation of the 2nd transition, measures 127–138, $F \rightarrow B\flat$

- **Coda,** measures 151–162, 4 variants of the 1st theme, $g_m \rightarrow D$, $B\flat \rightarrow F$, $f_m \rightarrow c_m$, $D_{-9} \rightarrow G$.

10 Discussion and farther problems

1. The paper suggests an algorithmic solution to Johnson's problem.
2. The approach can be adapted for finding rhythmic canons with several patterns not necessarily generated by augmentation (rhythmic fugues), which are also not necessarily constrained by the 'no gap' or 'no double beat' restrictions.
3. Motives can be regarded as generalized notes (note vectors).

   Not every generalized note can be fitted to an arbitrary sequence of generalized notes: Development of a theory of compatibility of note vectors, similar for harmony for single notes. This theory is rhythm-based, and therefore makes perspectives for a theory of rhythm.
4. Composition for note vectors is more restricted than for usual notes. Melodic expression requires respecting latent voice-leading, voice envelopes, etc.
5. Canons are mainy quasi-periodic (which is useful for composition). It is logical to enumerate elementary building blocks with various coupling profiles. Then constructing canons is reduced to manipulating with elementary blocks (like in puzzle-games).
6. Why not all canon lengths are available but 15, 30, 45, etc.