ON GENERALIZED INTERVALS AND TRANSFORMATIONS

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In an earlier paper, I proposed a certain approach to the formal study of pitch-class intervals and transformations.[1] There, I gave a number of reasons for developing such an approach; here I intend to demonstrate yet another of its values, that is, its susceptibility to wide generalization.

In the earlier work, we had a set S containing objects *s, t,* . . . of interest (the pitch-classes). With each ordered pair (s,t) of those objects, we associated an "interval from *s* to *t*." That interval, $i = \text{int}(s,t)$, was a certain sort of number. Specifically, the numbers involved, the "residues mod 12," could be manipulated among themselves in ways typical of mathematical groups. Intervals *i* and *j* could be *combined* by adding them mod 12. The *zero interval* was unique in that it combined with any interval *i* to form *i* itself. Intervals had *complements:* given *i*, the complementary interval $-i$ could be described as that unique interval which combined with *i* to form the zero interval.

The manipulation of intervals by such means obeyed certain laws which were taken for granted in the paper. For example, under the combining rule of addition mod 12, the combination of *i* with *j*, combined with *k,* produces the same result as does *i,* combined with the combination of *j* with *k.* That is, $(i+j)+k = i+(j+k)$ mod 12. This "Associative Law", together with another, the Commutative Law, implicitly assumed by the paper, were necessary for the formal inference of the results obtained. In mathematical language, we used the fact that the residues mod 12 form a "commutative group" under addition mod 12.

When we generalize the earlier work presently, we shall suppose that the generalized "intervals" under consideration are members of some commutative group. It therefore behooves us now to be more formal about that notion. By a *group,* we understand a set G, together with a *combination* rule which will be denoted here by the symbol *. This rule assigns, to any ordered pair (i,j) of members of G, a certain member of G which we shall denote $i*j$. The notion of i-star-j generalizes i-plus-j, i-times-j, i-plus-j-mod 12, etc. For G-cum-* to form a group, certain features must obtain. First, the combination must be *associative,* obeying the law discussed earlier: for every choice of i, j, and k, we must have $(i*j)*k = i*(j*k)$. Second, there must exist an *identity* element in G, that is an element e of G which combines with any given i to form that i: $e*i = i*e = i$. (In the group of residues under addition mod 12, the zero residue plays this role.) One can deduce that, given the associative law, there can be at most one such identity element in G; hence the one which is stipulated to exist is unique, and we can speak of "the" identity of the group. Third, given any i in G, there must exist a member i' of G which combines with i to form the identity: $i'*i = i*i' = e$. One can show that, given i, such an i' is unique. It is called the *inverse* of i in the group. (In the group of residues mod 12, the inverse of i is the complement $-i$ mod 12.)

Members i and j of a group are said to *commute* (with each other or as a pair) if $i*j = j*i$. A group G is called *commutative* or *non-commutative* depending on whether it is, or is not, the case that every i in G commutes with every j in G.

We are now ready to state a highly generalized form of the system developed in the earlier paper. Let G be some commutative group, to serve as a group of generalized "intervals." Let S be a set containing objects s, t, . . . of interest. Suppose that we can associate with each ordered pair (s,t) of those objects a certain "interval from s to t," a quantity i=int(s,t) which is a member of the group G. Suppose further that conditions (1), (2), and (3) following obtain.

(1) For any choice of objects s, t, and u, the interval from s to t, when combined in G with the interval from t to u, yields the interval from s to u. That is, int(s,t)*int(t,u) = int(s,u).

(2) For any choice of objects s and t, the interval from t to s is the inverse in G of the interval from s to t. That is, int(t,s) = int$(s,t)'$.

(3) Given any object s and any interval i in G, there is a unique

object t which lies the interval i from the given s, that is, which satisfies the equation $\text{int}(s,t) = i$.

These three conditions supposed, we can then demonstrate formally—that is, by deductive logic alone—results (4) through (7) following.

(4) For each interval i in G, the operation T_i of "transposition by i" can be defined on S. Given a specimen object s, its i-transpose is that (unique) object $t = T_i(s)$ which lies the interval i from the given s. That is, $T_i(s)$ is uniquely defined by the relation $\text{int}(s, T_i(s)) = i$.

(5) Given objects u and v, the operation I^{uv} of "inversion taking u to v" can be defined. Given a specimen object s, the inverted image $I^{uv}(s)$ of s is that (unique) object whose interval from u is the same as the interval from s to v: $\text{int}(u, I^{uv}(s)) = \text{int}(s,v)$.

(6) Let us consider various transformations X, Y, . . . which permute the objects of S. Let us call X "interval-preserving" if $\text{int}(X(s), X(t)) = \text{int}(s,t)$ for all choices of s and t. Let us call Y "interval-reversing" if $\text{int}(Y(s), Y(t)) = \text{int}(t,s)$ for all choices of s and t. Then theorems (a) and (b) following can be proved. (a) X is interval-preserving if and only if X is a transposition-operation. (b) Y is interval-reversing if and only if Y is an inversion-operation.

(7) The transpositions T_i and the inversions I^{uv} combine among themselves according to the laws (a) through (d) below.

(a) $T_i T_j = T_k$ where $k = i * j$.

(b) $T_i I^{uv} = I^{uw}$ where $w = T_i(v)$.

(c) $I^{uv} T_i = T_j I^{uv}$ where $j = i'$, the inverse of i.

(d) $I^{uv} I^{wx} = T_i$ where $i = \text{int}(w,v) * \text{int}(x,u)$.

(In the above formulas for (7), each operation-equation of the form $XY = Z$ is to be read: for any sample s, $X(Y(s)) = Z(s)$. That is, if you first perform Y on the sample object s, and then perform X on the object $Y(s)$, the net result will be the same as that of performing Z on s.)

To recapitulate: the results (4) through (7) can be inferred logically from the stipulations (1) through (3), together with the supposition that the intervals under consideration combine and have complements (inverses) in a commutative group. The method of inference follows exactly the formal arguments and proofs elaborated in the earlier article, to which the curious reader is referred. Such a reader will note, in particular, that the commutativity of G is *essential* to the logic of the proofs for (6) and (7); this is a crucial point to which we shall return later. If G is not commutative, our intuition concerning the operations (still) defined by (4) and (5) is not reliable; it may, for instance, happen that I^{uv} and I^{vu} are different operations in that case.

Some examples will illustrate the generality of the (commutative) system we have developed, as well as some of its problems and limitations. For Example 1, let S be the family of all durations. We can imagine the quantities $s, t, \ldots$ to be positive real numbers, telling us how long the durations are as multiples of some fixed unit of time. Let us consider the "interval from s to t" here to be the quotient t/s. Taking that quotient as int(s,t), we see that such intervals are themselves positive real numbers. That family of numbers forms a commutative group under multiplication. (Multiplication is associative and commutative, 1 is the identity, and the group inverse of i is its reciprocal $1/i$.) One can also verify that stipulations (1) through (3) obtain in this model. Specifically, int(s,t)*int(t,u) = (t/s) (u/t) = u/s = int(s,u); thus stipulation (1) obtains. And s/t, the interval from t to s, is the reciprocal (inverse in G) of t/s, the interval from s to t; thus (2) obtains. Finally, given a duration s and an interval i, then any duration t which lies the interval i from s must satisfy int$(s,t) = i$, that is $t/s = i$. It is clear that the duration $t = i$-times-s is a unique such duration; hence (3) obtains.

We can now conclude, without further ado, that all of the results (4) through (7) obtain in this model. Via (4), we can see that transposing s by i amounts to multiplying s by i : $T_i(s) = is$. Given u and v, I^{uv} takes a sample duration s to the inverted duration $I(s)$ defined by formula (5): $I(s)/u = v/s$. (That is, $I(s)$ is j times as long or as short as u, where v is j times as long or as short as the specimen s.) From the latter relationship we can compute $I(s) = uv/s$. Theorem (6) tells us that these transpositions and inversions are respectively the interval-preserving and the interval-reversing operations on durations, in the present sense of "interval." And formulas (7) show us how to manipulate the labels attached to those operations when we combine them.

While the system just studied behaves well formally, it is problematical as a model for musical rhythm in all contexts. Let us leave aside the fact that S contains indefinitely long and indefinitely short durations; let us also ignore the fact that S contains an infinitude of even those durations of potential practical musical interest; problems analogous to these have been confronted effectively in a variety of pitch-systems. Let us rather concern ourselves here with the fact that our perception of relations among musical durations is at least as much additive as multiplicative, particularly in the foreground of rhythmic textures. That is, given durations s and t, we often perceive t as so much longer (or shorter) than s by a certain *difference* t-s, rather than so many times as long (or as short) by a certain *factor* t/s.

For Example 2, let us then try to construct a model in our system, using the same family S of durations, but now taking int(s,t) to be the difference t-s of the durations t and s. E.g. int(s,t) = 3, -2 or 0 if t is respectively 3 time-units longer than, 2 units shorter than, or the same length as s. Our group G will be all real numbers, combined by addition. That is, $i*j$ will mean $i+j$. (It is easily verified that G-cum-* is indeed a commutative group.) Stipulation (1) obtains: given s, t and u, then int(s,t)*int(t,u) = $(t-s)+(u-t)$ = $u-s$ = int(s,u) as desired. Stipulation (2) also obtains: given s and t, then int(t,s) = $s-t$ = $-(t-s)$ = int(s,t)$'$ as desired. But stipulation (3) does *not* obtain for this model: given a duration s and an interval i, we can not always find a duration t satisfying $t-s = i$. Suppose, for instance, that s is 2 units, and i is the interval -5 (units); we are then seeking a duration t which is 5 units shorter than 2 units of time, i.e. which lasts (measurably) 3 units less than no time at all. Algebraically, since we demand t-2 = -5, we would have to have t = -3. Our present model makes no sense of this concept; the members of S are supposed to measure only positive quantities of time. Hence our general system is not realized by this model. To realize the system, one sees that we will have to be able to attribute a meaning to "negative durations" as objects in the family S. This can be done by adopting any of a variety of conventions. Any such convention, though, would involve changing the nature of S and/or G.

Rather than pursuing such possibilities here, let us instead investigate another, related, model as our Example 3. We take S here to be the family, not of durations, but of time-points. We shall label each member s of S by a real number, positive, negative or zero, measuring the distance of s from an arbitrary zero time-point, in units of time. We take the interval from s to t to be the numerical difference $t-s$; this interval is then, for example, 3, -2 or 0 respectively as time-point t occurs 3 units later than, 2 units earlier than, or exactly at the same moment as time-point s. In this model, durations arise not as objects but as intervals between objects. A negative duration, as int(s,t), simply means here that t occurs before s. G is taken as the group of all real numbers, combining under addition. It is easily verified, as in the work for Example 2, that stipulations (1) and (2) obtain here. Stipulation (3), which failed for Example 2, is now valid for Example 3. Given a time-point s and i units of time, we can find a unique time-point t satisfying $t-s = i$. If i is positive, t occurs i units after s; if i is negative, t occurs $(-i)$ units before s. Without further ado, we can now conclude that all the results of (4) through (7) are valid for this system. It should be noted that int(s,t) is independent of the "arbitrary zero time-point" selected earlier; int(s,t) still does depend, however, on the unit by which time is being measured (e.g. the "beat," the "measure," the "whole note," a second, a centimeter of tape, etc.)

Other rhythmic models instancing the general system can be studied. An important class of such models is obtained by combining one of the models already studied (including Example 2) with notions of "equivalence" among certain classes of durations and intervals, in a manner analogous to the formalism induced by "octave equivalence" among classes of pitches.[2] Another important family of models involves quantization of the sets *S* and groups *G*, so as to involve discrete or even finite, rather than continuous, families of objects and intervals.[3] Most such reduced models instance the general system: one obtains a (reduced) family of durations, duration-classes, time-points or time-point classes, and a (reduced) commutative group of formal intervals spanned by ordered pairs of those objects, such that stipulations (1), (2) and (3) obtain. Without further ado, results (4) through (7) will be valid.

For a more complex and less familiar model exemplifying the general system, I propose to investigate intervals between "spans." By a "span," I shall mean an ordered pair $s = (s_0, s_1)$ of real numbers, positive, negative or zero, such that s_0 is less than s_1. The lesser number s_0 will be called the "beginning" of the span *s*, and we shall write $s_0 = BGN(s)$. Similarly, s_1, the "end" of *s*, can be denoted $END(s)$. We shall also have occasion to consider the midpoint of *s*, $MID(s)$. This is the number halfway between $BGN(s)$ and $END(s)$. The length of *s* will be denoted by $LEN(s)$, that is the difference $END(s)\text{-}BGN(s)$.

Such numerical spans could model various musical phenomena: s_0 and s_1 might be time-points articulating the beginning and end of a segment of time. Or, considering s_0 and s_1 as frequencies, they could articulate the low and high extremes of a certain band-limited frequency event. Or, still as frequencies, they could represent the low and high half-power points of a certain noise or filter characteristic. Or, now as pitches, they could represent the lowest and highest notes present in a certain cluster. Or, again as pitches, they could articulate the extreme notes of an active register in a certain piece at a certain time. And so on. (In each case, we would have to specify a convention for attributing meaning to negative values for s_0, or for both s_0 and s_1. In each case, this could be done.)

For Example 4 we shall take, as the "interval between span *s* and span *t*," the pair of numbers (x,a), where x is the quotient of the lengths of the spans and a is the directed distance between their midpoints. That is, $x = LEN(t)/LEN(s)$ and $a = MID(t)\text{-}MID(s)$. Since the lengths of the spans are positive, x will be a positive real number; a will be a real number.

The group *G* for Example 4 will comprise all ordered pairs (x,a) such that x is a positive real number and a is a real number. The group combination will be defined as $(x,a)*(y,b) = (xy, a+b)$. It is straightforward to verify that this combination is associative and commutative, that (1,0) is an identity for the system, and that $(1/x, -a)$ combines with (x, a) to form (1,0). So G is a commutative group.

It can be shown that stipulations (1) through (3) of the general system obtain for this model. Now, without further ado, we can conclude that the results of (4) through (7) will also obtain. To transpose the span s by the interval $i = (x,a)$, we first expand or contract s about its midpoint by a factor of x, and then displace the resulting span rigidly by a units. Or, equivalently, we can first displace s rigidly by a units, and then expand or contract the resultant span itself about its midpoint by a factor of x. Given spans u and v, and writing I for I^{uv}, formula (5) tells us how to compute the span $I(s)$, given the span s. First we compute int(s,v): this is the pair (x,a) such that $x = LEN(v)/LEN(s)$ and $a = MID(v)\text{-}MID(s)$. Formula (5) tells us that this (x,a) will also be int$(u,I(s))$. Hence the above x also $= LEN(I(s))/LEN(u)$, and the above a also $= MID(I(s))\text{-}MID(u)$. Working out the equalities, we can derive $LEN(I(s)) = LEN(u)LEN(v)/LEN(s)$, and $MID(I(s)) = MID(u) + MID(v)\text{-}MID(s)$. The latter equations compute the length and the midpoint of the span $I(s)$ in terms of the known lengths and midpoints of the spans u, v, and s; knowing the midpoint and the length of the span $I(s)$ we can of course construct its beginning and its end, i.e. we know what $I(s)$ is.

The model of Example 4, as a means of conceptualizing and manipulating intervals and transformations involving spans, enjoys decided advantages because of the validity of (6) and (7). On the other hand, the model has certain problematic aspects qua model. Among these is the significance it attaches to the midpoint of each span. It is clear that there are many musical situations in which one would not care to assign such a priori functional significance to the midpoint of an articulated span of, say, pitches or time-points. And, in fact, other systems of span-intervals can be developed which do not attach such formal weight to the midpoints of spans, and which correspond, for many applications, to more "musical" ways of deforming spans, one into another. Such systems, however, sacrifice the algebraic advantages enjoyed by Example 4, and the considerable clarity and power that go with those advantages.

For instance, given spans s and t, there exists a unique numerical transformation of a sort called "projective" which transforms s_0, s_1, and a number between into, respectively, t_0, t_1, and a number between. If we call this projective transformation P, we can write symbolically $P(s) = t$. If we follow P by another projective transformation Q of the sort under consideration, the resulting composite transformation will itself be of the desired type; further, such transformations form a group under this rule of combination; that is, $P*Q$ means

P-followed-by-Q. Let us try defining a member P of the group to be "int(s,t)" when P is the unique transformation of the sort under consideration satisfying $P(s) = t$. If we so define int(s,t), then stipulations (1) through (3) of the general system are satisfied. That being so, (4) and (5) will still provide formal definitions for "transposition" and "inversion" operations. But very little of (6) and (7) will obtain. In fact, it is not even the case here that I^{uv} (as defined by (5)) is the same operation as I^{vu}: "inversion taking u to v" is not the same operation as "inversion taking v to u." The trouble is that the group under consideration is *non*-commutative. As was mentioned earlier, commutativity in the group of intervals is essential, within the general system, if one is to infer (6) and (7) from (1) through (3). Thus, though projective transformations are in some respects more plausible for many musical situations than are the "transpositions" of Example 4, the projective system sacrifices a good deal of power and intuitive clarity in exchange.

The projective system above does exemplify an interesting attitude, though. It starts with the notion of transforming its objects, one into another, and then defines, as the interval from s to t, a certain transformation, unique of its sort, which carries s to t. By this means, transposition operations are actually conceived as defining intervals, rather than vice-versa. This is a suggestive notion. To return to the general (commutative) system: it is often useful to think of an interval i not as an abstract directed "distance" from s to t, but rather as a label for the corresponding transposition operation T_i, a unique operation of its kind which "moves s to t." The transposition operations form a group which can be identified with the group of intervals via formula (7a). (Mathematicians call the groups "isomorphic.") The reader who wishes to explore further the idea of considering transposition operations themselves as "intervals" will find discussion of related matter in another of my earlier articles.[4]

NOTES

1. David Lewin, "A Label-Free Development for 12-Pitch-Class Systems," *Journal of Music Theory* 21 (1977), pp. 29–48.
2. Instances of such models have been developed, e.g., by Stockhausen and Babbitt, in "multiplicative" and "additive" contexts respectively. See Karlheinz Stockhausen, ". . . how time passes . . . ," *Die Reihe* 3, pp. 10–40, and Milton Babbitt, "Twelve-Tone Rhythmic Structure and the Electronic Medium," *Perspectives of New Music* 1 (1962), pp. 49–79.
3. Babbitt's model, in "Twelve-tone Rhythmic Structure," has this feature.
4. David Lewin, "Forte's Interval Vector, My Interval Function, and Regener's Common-Note Function," *Journal of Music Theory* 21 (1977), pp. 194–237. The discussion in Parts 4 and 5 of that article is particularly to the point.